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The positive P representation and the laser equations

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Abstract. The drift in the Fokker-Planck equation in the positive P representation describing a gas laser is shown to have a continuum of stable steady states. Only a small subset is physically acceptable. It is demonstrated that the diffusion does manage to average over the unphysical steady states to give correct correlation properties in the lasing region. However, in the chaotic region the positive P representation equations have additional instabilities. As a consequence the positive P representation in the chaotic region leads to behaviour either different from known solutions or breaks down.

1. Introduction

The dynamics in quantum non-equilibrium statistical mechanics can be expressed through a master equation [1, 2] for the density matrix operator. Frequently this equation is converted into one for a c-number function, known as a quasi-probability distribution which is not a distribution in the sense of classical probability theory. It is often the case that the quasi-probability distributions satisfy a Fokker-Planck equation [3] to a good approximation. The non-classical nature of the distributions is caused by diffusion matrices associated with the Fokker-Planck equation which are not uniformly positive definite throughout phase space. In order to be able to use the classical concepts of probability distributions, Drummond and Gardiner [4] proposed extending the physical phase space by incorporating non-physical degrees of freedom. In fact the dimension of the resulting phase space is twice that of the original space. Such distributions would allow 'classical' probability distributions to describe non-classical phenomena [5] such as anti-bunching. We will now outline a way of introducing the positive P representation. If $\mathcal{S}=\{O_i, O_i^\dagger\}$ is a complete set of operators for a system (in the sense that any other operators can be expressed as a polynomial of these operators) then the characteristic function [6] $\chi_{\mathcal{S}}(\lambda_j, \lambda_j')$ associated with $\mathcal{S}$ is

$$\chi_{\mathcal{S}}=\mathrm{Tr}\left(\rho \prod_{j=1}^{n} \exp(\mathrm{i}\lambda_j O_j^\dagger) \prod_{j'=1}^{n} \exp(\mathrm{i}\lambda_{j'} O_{j'})\right) \tag{1}$$

and ρ is the density matrix. A function $P_{\mathcal{S}}(\{\alpha_j, \alpha_j'\})$ can be associated with $\chi_{\mathcal{S}}$, namely

$$\chi_{\mathcal{S}}(\{\lambda_j, \lambda_{j'}\})=\int \prod_{i=1}^{n} \mathrm{d}^2\alpha_i \prod_{i'=1}^{n} \mathrm{d}^2\alpha'_{i'} \exp\left(\mathrm{i} \sum_{i=1}^{n} \lambda_j\alpha_j\right) \exp\left(\mathrm{i} \sum_{k=1}^{n} \lambda'_k\alpha'_k\right) P_{\mathcal{S}}(\{\alpha_i, \alpha'_i\}). \tag{2}$$

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Here λ_j and λ_j' are independent complex numbers for all j. There is no explicit transform which expresses $P_{\mathscr{S}}$ in terms of $\chi_{\mathscr{S}}$. In fact, the existence or the uniqueness of such a P cannot be proved in general. For the case when $\mathscr{S}=\{a, a^+\}$, and a and a^+ are the annihilation and creation operators for the harmonic oscillator, it can be shown that there exists a suitable $P_{\mathscr{S}}$. A possible $P_{\mathscr{S}}(\alpha, \alpha')$ has the form [2]

$$P_{\mathscr{S}}(\alpha, \alpha')=\frac{1}{4\pi^2}\exp(-\tfrac{1}{4}|\alpha'-\alpha^*|^2)\langle\tfrac{1}{2}(\alpha'+\alpha^*)|\rho|\tfrac{1}{2}(\alpha'+\alpha^*)\rangle. \tag{3}$$

The state $|\beta\rangle$ is such that

$$a|\beta\rangle=\beta|\beta\rangle. \tag{4}$$

However, when $\mathscr{S}$ contains atomic operators no such simple form for $P_{\mathscr{S}}$ is known. The master equation for ρ implies in general a partial differential equation with derivatives of arbitrarily high order for $\chi_{\mathscr{S}}$. If a system size expansion [1] is valid this reduces to a second-order one in $\{\lambda_i, \lambda_i'\}$. In turn this approximate equation can be satisfied if $P_{\mathscr{S}}$ satisfies a Fokker–Planck equation (provided $P_{\mathscr{S}}$ falls off sufficiently fast at infinity to allow integration by parts). This is a sufficient condition, but not a necessary one, and there is no guarantee that a self-consistent solution for P exists.

In $P_{\mathscr{S}}(\{\alpha_i, \alpha_i'\})$ the α_i and α_i' are independent complex numbers for all i. This is the source of the doubling of the dimension of the apparent phase space. The above discussion is formal. A rigorous derivation of a positive P representation has only been given for theories containing just photon operators. Little non-perturbative analysis of the positive P equations for the laser (where atomic variables are present) has been done. This will be given here.

A standard master equation [7] for a gas laser will be solved in the Fokker–Planck approximation for $P_{\mathscr{S}}$. The drift in the Fokker–Planck equation for $P_{\mathscr{S}}$ is found to have a continuum of steady states whose modulus is not a fixed constant (as found in the Wigner representation [8]). For this reason the majority of the steady states are unphysical.

In the lasing region the phase diffusion is adequate to average over these unphysical steady states to give correct properties. In the chaotic region the eigenvalues of the stability matrix around the steady states indicate instabilities in unphysical directions such as the imaginary part of the inversion. This instability grows to give severe lack of conjugacy in variables which represent conjugate operators; owing to this the positive P representation has difficulties in the chaotic region and leads to a picture quite different from that which is expected and known. Moreover for large enough (but moderate) noise the positive P picture breaks down.

In § 2 the laser master equation and the associated Ito–Langevin stochastic differential equations [2, 9] are presented for the positive P representation.

2. The laser stochastic differential equation

The usual master equation for the laser [7] is

$$\mathrm{d}\rho/\mathrm{d}t=\frac{-\mathrm{i}}{\hbar}[H, \rho]+([a\rho, a^+]+[a, \rho a^+])+\Lambda_A\rho \tag{5}$$

where the Hamiltonian H and the Liouvillian Λ_A are given by

$$H=\mathrm{i}gh\sum_{\mu=1}^{N}[\exp(-\mathrm{i}k\cdot x_\mu)a^+\sigma_\mu^- - \exp(\mathrm{i}k\cdot x_\mu)a\sigma_\mu^+] \tag{6}$$

$$\Lambda_A\rho = \tfrac{1}{2}\sum_{\mu=1}^{N}\{\gamma_u([\sigma_\mu^+, \rho\sigma_\mu^-]+[\sigma_\mu^+\rho, \sigma_\mu^-])+\gamma_\|([\sigma_\mu^-, \rho\sigma_\mu^+]+[\sigma_\mu^-\rho, \sigma_\mu^+])$$

$$+\gamma_0([\sigma_\mu^z, \rho\sigma_\mu^z]+[\sigma_\mu^z\rho, \sigma_\mu^z])\}. \tag{7}$$

The coupling constant g is $(2\pi\omega\mu^2/\hbar V)^{1/2}$ where μ is the atomic dipole moment, V the cavity volume and ω is the resonance frequency of the atoms and of the field mode in tune with it; ρ is the density matrix. a and a^+ are annihilation and creation operators for the field mode in a ring cavity. The atoms are modelled as having only two levels and $\{\sigma_\nu^\pm, \sigma_\nu^z\}$ are the Pauli matrices associated with the νth atom. κ is the cavity damping rate, $\gamma_\|$ and γ_0 are atomic population and collision-induced polarisation decay rates and γ_u is related to a pumping rate. In order to have lasing it is necessary that $\gamma_u > \gamma_\|$. Moreover we will consider the situation where $\gamma_0 \gg \gamma_u$. After somewhat lengthy analysis the Ito equations associated with the positive P distribution can be found in terms of intensive variables.

Z is a complex intensive variable proportional to the inversion; X_1, X_2, Y_1 and Y_2 are independent complex intensive variables 'proportional to' a, a^+,

$$\sum_\mu \sigma_\mu^- \qquad \text{and} \qquad \sum_\mu \sigma_\mu^+.$$

The equations generate expectation values of normal ordered operator products. They are as follows:

$$\mathrm{d}X_1 = \sigma(Y_1 - X_1)\,\mathrm{d}\tau \tag{8}$$

$$\mathrm{d}X_2 = \sigma(Y_2 - X_2)\,\mathrm{d}\tau \tag{9}$$

$$\mathrm{d}Y_1 = (-Y_1 + X_1 Z)\,\mathrm{d}\tau + 2^{3/2}\varepsilon\sum_{\nu=1}^{3} \mathrm{d}_{1\nu}\,\mathrm{d}W_\nu \tag{10}$$

$$\mathrm{d}Y_2 = (-Y_2 + X_2 Z)\,\mathrm{d}\tau + 2^{3/2}\varepsilon\sum_{\nu=1}^{3} \mathrm{d}_{3\nu}\,\mathrm{d}W_\nu \tag{11}$$

$$\mathrm{d}Z = b(r - Z)\,\mathrm{d}\tau - (Y_1X_2 + Y_2X_1)\,\mathrm{d}\tau + 2\varepsilon\sum_{\nu=1}^{3} \mathrm{d}_{2\nu}\,\mathrm{d}W_\nu \tag{12}$$

where

$$b = \gamma_\|/\gamma_1 \qquad \sigma = \kappa/\gamma_\perp$$

$$r = 2C(\gamma_u - \gamma_\|)/(\gamma_u + \gamma_\|)$$

$$\tau = \gamma_\perp t$$

$$\gamma_\perp = 2\gamma_0 + \tfrac{1}{2}\gamma_\|$$

$$\varepsilon = (cb/8\sigma n_0)^{1/2}$$

$$n_0 = \gamma_\| N/8\kappa C$$

$$C = g^2N/2\kappa\gamma_\perp$$

and

$$dd^T = D \tag{13}$$

with D a symmetric matrix defined by

$$D=\begin{pmatrix} \frac{4X_1Y_1}{C} & -\frac{2b}{C}Y_1\left(1+\frac{r}{2C}\right) & \frac{4}{C}(Z+2C) \\ & \frac{2b}{C}\left(4C-\frac{rZ}{C}-\frac{2}{b}(X_1Y_2+Y_1X_2)\right) & -\frac{2b}{C}Y_2\left(1+\frac{r}{2C}\right) \\ & & \frac{4X_2Y_2}{C} \end{pmatrix}. \tag{14}$$

There is no unique solution for d; d, for example can be multiplied by an orthogonal matrix. We have tried two decompositions; these give similar results in regions where both are non-singular.

The two decompositions are best given for a general (3×3) matrix

$$\begin{pmatrix} \mu & \alpha & \beta \\ \alpha & \nu & \delta \\ \beta & \delta & \lambda \end{pmatrix}.$$

The first expression for d is

$$\begin{pmatrix} (\bar{\mu})^{1/2}\left(1+\frac{\bar{\beta}}{\bar{\mu}}\right) & 0 & \mathrm{i}\left(\frac{\bar{\beta}^2}{\bar{\mu}}+\bar{\lambda}\right)^{1/2} \\ \frac{\bar{\alpha}}{(\bar{\mu})^{1/2}} & \left[\left(\bar{\delta}-\frac{\bar{\alpha}\bar{\beta}}{\bar{\mu}}\right)^2\left(\bar{\lambda}+\frac{\bar{\beta}^2}{\bar{\mu}}\right)^{-1}+\left(\nu-\frac{\bar{\alpha}^2}{\bar{\mu}}\right)\right]^{1/2} & -\mathrm{i}\left(\bar{\delta}-\frac{\bar{\alpha}\bar{\beta}}{\bar{\mu}}\right)\left(\bar{\lambda}+\frac{\bar{\beta}^2}{\bar{\mu}}\right)^{-1/2} \\ (\bar{\mu})^{1/2}\left(1-\frac{\bar{\beta}}{\bar{\mu}}\right) & 0 & -\mathrm{i}\left(\frac{\bar{\beta}^2}{\bar{\mu}}+\bar{\lambda}\right)^{1/2} \end{pmatrix} \tag{15}$$

where

$$\begin{aligned} \bar{\mu} &= \tfrac{1}{4}(\lambda+\mu)+\tfrac{1}{2}\beta \\ \bar{\lambda} &= \tfrac{1}{2}\beta-\tfrac{1}{4}(\lambda+\mu) \\ \bar{\beta} &= \tfrac{1}{4}(\mu-\lambda) \\ \bar{\alpha} &= \tfrac{1}{2}(\alpha+\delta) \\ \bar{\delta} &= \tfrac{1}{2}(\alpha-\delta). \end{aligned} \tag{16}$$

The second Cholesky decomposition has the following expression for d:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha/\mu & 1 & 0 \\ \beta/\mu & (\mu\delta-\alpha\beta)/(\mu\nu-\alpha^2) & 1 \end{pmatrix} \times \begin{pmatrix} \mu^{1/2} & & \\ & (\nu-\alpha^2/\mu)^{1/2} & \\ & & \{[\lambda\mu-\beta^2-(\mu\delta-\alpha\beta)^2/(\mu\nu-\alpha^2)]\mu^{-1}\}^{1/2} \end{pmatrix}. \tag{17}$$

Clearly this is singular when μ is very small and so would be unsuitable near the laser threshold.

3. The deterministic case

We shall examine the steady states of (8)-(12) in the absence of noise and for r non-zero. These states for $r > 1$ are given by

$$X_1 = Y_1 = X_1^{(0)} = |X_1^{(0)}| \, e^{i\phi} \tag{18}$$

$$X_2 = Y_2 = X_2^{(0)} = |X_2^{(0)}| \, e^{-i\phi} \tag{19}$$

$$|X_1^{(0)}||X_2^{(0)}| = \tfrac{1}{2}b(r-1). \tag{20}$$

Similarly for $r < 1$ we have

$$X_1 = Y_1 = X_1^{(0)} = |X_1^{(0)}| \, e^{i\phi} \tag{21}$$

$$X_2 = Y_2 = X_2^{(0)} = -|X_2^{(0)}| \, e^{-i\phi} \tag{22}$$

$$|X_1^{(0)}||X_2^{(0)}| = \tfrac{1}{2}b(1-r). \tag{23}$$

In addition there is a state

$$X_1 = X_2 = 0 = Y_1 = Y_2 \tag{24}$$

$$Z = r. \tag{25}$$

The steady states for which conjugacy does not hold are unphysical. A linearised stability analysis can be done for any of these states (and the details can be found in the appendix). For (18)-(23) the associated eigenvalues (which each appear twice) satisfy

$$\lambda(\lambda+\sigma+1)[\lambda(\lambda+\sigma+1)(\lambda+b)+b(r-1)(\lambda+2\sigma)]=0. \tag{26}$$

The cubic in the above brackets appears in the analysis of the real Lorenz equations [10-12]. Consequently there is a subcritical Hopf bifurcation [10-12] to chaos for our system at the same value of r (>1) for all the steady states. It can be shown that (21)-(23) define unstable states, while for $0 < r < 1$ the states of (24) and (25) are stable.

We will now examine some general properties of the time-dependent solutions of the positive P deterministic equations (8)-(12)). The variable

$$W = X_2 Y_1 - Y_2 X_1 \tag{27}$$

satisfies

$$\dot{W} = -(1+\sigma) W. \tag{28}$$

The vanishing of W is a necessary (but not sufficient) condition for conjugacy. Clearly

$$W(t) = W(0) \exp[-(1+\sigma)t] \to 0 \qquad \text{as } t \to \infty \tag{29}$$

a very rapid decay indeed when $\sigma = 10$. We would expect the behaviour of W to be robust in the presence of noise.

Now

$$\frac{d}{dt} \log \frac{X_1}{X_2} = \frac{\sigma}{X_1 X_2} W. \tag{30}$$

Similarly

$$\frac{d}{dt} \log \frac{Y_1}{Y_2} = \frac{-\sigma}{Y_1 Y_2} W. \tag{31}$$

Since W vanishes asymptotically (30) and (31) imply

$$X_1/X_2 = Y_1/Y_2 = \alpha \qquad t \to \infty. \tag{32}$$

Again, this asymptotic value should be 'reached' quickly. Here α is a complex constant. In fact from (30)

$$\frac{X_1(t)}{X_2(t)} = \frac{X_1(0)}{X_2(0)} \exp\left(\sigma(X_2(0)Y_1(0) - X_1(0)Y_2(0)) \int_0^t \mathrm{d}t' \frac{\exp[-(1+\sigma)t']}{X_1(t')X_2(t')} \right). \tag{33}$$

α can be calculated explicitly in two cases, which we now give. If

$$\begin{aligned} X_2(0) &= (X_1(0))^* = \chi\, \mathrm{e}^{-\mathrm{i}\phi} \\ Y_2(0) &= (Y_1(0))^* = \psi\, \mathrm{e}^{-\mathrm{i}\phi} \end{aligned} \tag{34}$$

with $Z(0)$, χ and ψ real, then

$$\frac{X_1(t)}{X_2(t)} = \frac{X_1(0)}{X_2(0)} = \alpha = \mathrm{e}^{2\mathrm{i}\phi}. \tag{35}$$

Also, if

$$\begin{aligned} X_2(0) &= (X_1(0))^* = \chi\, \mathrm{e}^{-\mathrm{i}\phi_x} \\ Y_2(0) &= (Y_1(0))^* = \psi\, \mathrm{e}^{-\mathrm{i}\phi_y} \end{aligned} \tag{36}$$

with $Z(0)$, χ and ψ real, then

$$\frac{X_1(t)}{X_2(t)} = \mathrm{e}^{2\mathrm{i}\phi_x} \exp\left(-\mathrm{i}\sigma 2\chi\psi \sin(\phi_x - \phi_y) \int_0^{t'} \mathrm{d}t' |X_1(t')|^{-2} \exp[-(\sigma+1)t'] \right). \tag{37}$$

Clearly $X_1(t)/X_2(t)$ is unimodular. Hence

$$\alpha = \mathrm{e}^{2\mathrm{i}\phi} \qquad \text{as} \qquad t \to \infty \tag{38}$$

with

$$\phi = \phi_x - \sigma\chi\psi \sin(\phi_x - \phi_y) \int_0^{\infty} \mathrm{d}t' |X_1(t')|^{-2} \exp[-(\sigma+1)t']. \tag{39}$$

Equations (35) and (37) are only necessary conditions for conjugacy. However since

$$(\mathrm{d}/\mathrm{d}t)(X_1 - X_2^*) = -\sigma(X_1 - X_2^*) + \sigma(Y_1 - Y_2^*) \tag{40}$$

$$(\mathrm{d}/\mathrm{d}t)(Y_1 - Y_2^*) = -(Y_1 - Y_2^*) + (X_1 - X_2^*)Z \tag{41}$$

$$(\mathrm{d}/\mathrm{d}t)Z = -b(Z - r) - (X_1 Y_2 + X_2 Y_1) \tag{42}$$

evolution from the initial conditions of (34) and (36) preserve conjugacy when there is rigorously no noise to arbitrary precision. In order to examine the stability of the conjugacy relationship it is useful to first obtain equations which have incorporated the asymptotic conditions given by (32) and the vanishing of W. The time evolution asymptotically, i.e. on the attractor, is given by

$$\dot{X} = -\sigma(X - Y) \tag{43}$$

$$\dot{Y} = -Y + XZ \tag{44}$$

$$\dot{Z} = -b(Z - r) - XY \tag{45}$$

where

$$X = (2/\bar{\alpha})^{1/2} X_1 \qquad Y = (2/\bar{\alpha})^{1/2} Y_1. \tag{46), (47}$$

X, Y and Z are in general complex. Conjugacy would require these variables to be real. In order to examine small deviations from conjugacy we can write

$$X = X_r + \mathrm{i}\delta_x \tag{48}$$

$$Y = Y_r + \mathrm{i}\delta_y \tag{49}$$

$$Z = Z_r + \mathrm{i}\delta_z \tag{50}$$

where X_r, Y_r, Z_r, δ_x, δ_y and δ_z are real. Equations (43)-(45) imply

$$\begin{pmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \dot{\delta}_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ Z_r & -1 & X_r \\ -Y_r & -X_r & -b \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \end{pmatrix}. \tag{51}$$

The evolution of X_r, Y_r and Z_r is determined from the Lorenz equations (43)-(45). In the chaotic region the matrix in (51) has eigenvalues with positive real part on most points of a chaotic trajectory. In contrast, for the lasing region, the eigenvalues have negative real part. This indicates that conjugacy is unlikely to be maintained in the chaotic region in the presence of noise. Further support for this is found from constructing an eigenvector $\boldsymbol{e}$ corresponding to an eigenvalue λ satisfying (26) but with $\lambda \neq \{0, -(\sigma+1)\}$:

$$\boldsymbol{e} = \begin{pmatrix} a \\ 0 \\ (1+\lambda\sigma^{-1})a \\ 0 \\ -\dfrac{2a[1+(\sigma+\lambda)\sigma^{-1}]}{\lambda+b} X_+^{(0)} \end{pmatrix} \tag{52}$$

($\boldsymbol{e}$ needs to be normalised by fixing the constant a). The basis used for this eigenvector is described in the appendix. The fifth component denotes a projection onto the imaginary Z direction. In the chaotic region there are eigenvalues with positive real part and so a perturbation with an imaginary component in the Im Z 'direction' will tend to grow and as a result conjugacy will break down.

4. The general (noisy) case

The full set of equations (8)-(12) will be solved numerically using a lowest-order Euler algorithm [2]. From our (largely analytic) study of the deterministic case we expect to find instabilities in the chaotic region and consequent difficulties for the positive P picture in the presence of noise. Both the properties of individual trajectories and the effect of averaging over contributions from different trajectories are discussed below. Our numerical calculations indicate the following.

(i) W falls to the noise floor.

(ii) In the lasing region (e.g. $r = 10$ and 16) conjugacy is preserved even when there is noise. In particular (X_1/X_2), (Y_1/Y_2) are approximately constant with modulus of near one. Im Z remains small and fluctuating.

(iii) Over sufficiently long times, and for both conjugate and non-conjugate initial conditions, we find uniform phase diffusion in the lasing region; moreover conjugacy is recovered for the individual solution trajectories even if initial conditions are non-conjugate.

(iv) In the chaotic region the solution trajectories move away from conjugacy. This is illustrated by four cases with $r=28$, $\sigma=10$, $b=\frac{8}{3}$. $C=50$.

(a) A deterministic situation with conjugate initial conditions

$$\begin{aligned} &X_1 = X_2 = 0.63 \\ &Y_1 = Y_2 = 0.37 \\ &Z = 9.5. \end{aligned} \tag{53}$$

The quantities calculated are

$$|X_1 - X_2^*| \qquad |X_1 - X_2^*|/(|X_1| + |X_2|)$$

$$\text{Re}\, Z \qquad \text{Im}\, Z \qquad \log|W| \qquad \arg X_1/X_2 \qquad |X_1|/|X_2|.$$

(These same quantities will be evaluated for the cases (b), (c) and (d) below.) Exact conjugacy would require

$$|X_1 - X_2^*| = \frac{|X_1 - X_2^*|}{|X_1| + |X_2|} = \text{Im}\, Z = 0. \tag{54}$$

Moreover for $\sigma = 10$ from (29), (31) and (32) we would expect (even in the presence of moderate noise) that the evolution would rapidly show

$$\begin{aligned} &W \sim 0 \qquad \frac{|X_1|}{|X_2|} \sim 1 \\ &\arg(X_1/X_2) \sim \text{constant}. \end{aligned} \tag{55}$$

Figures 1–4 show a departure from conjugacy. This instability is started off by the (very small) round-off noise. The measure of non-conjugacy is bounded but macroscopic.

Figures 5–7 show the expected behaviour for $\log|W|$, $|X_1/X_2|$ and $\arg X_1/X_2$.

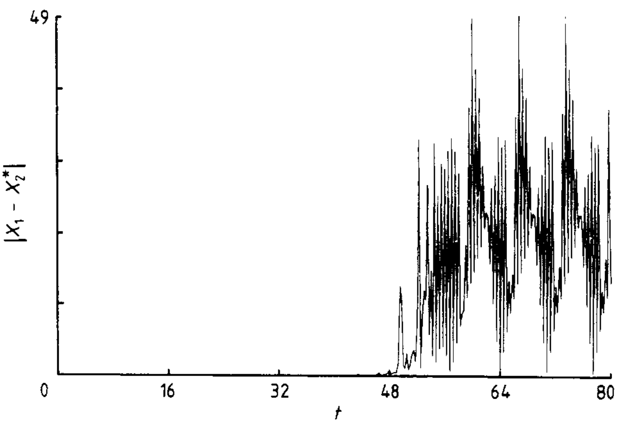

Figure 1. $|X_1 - X_2^*|$ against time for conjugate initial conditions and $\varepsilon = 0$.

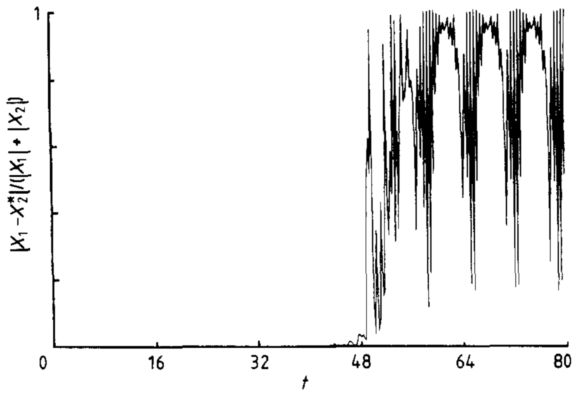

Figure 2. $|X_1 - X_2^*|/(|X_1|+|X_2|)$ against time for conjugate initial conditions and $\varepsilon = 0$.

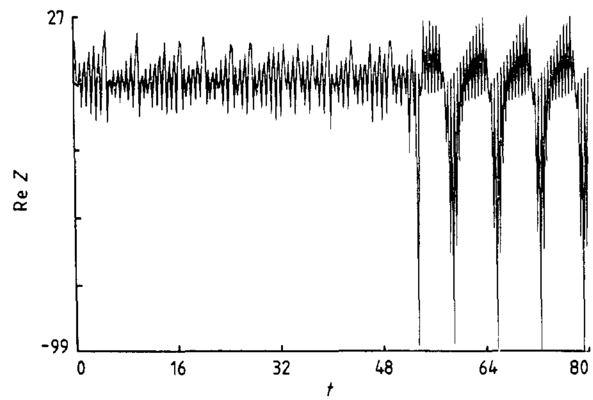

Figure 3. Re Z against time for conjugate initial conditions and $\varepsilon = 0$.

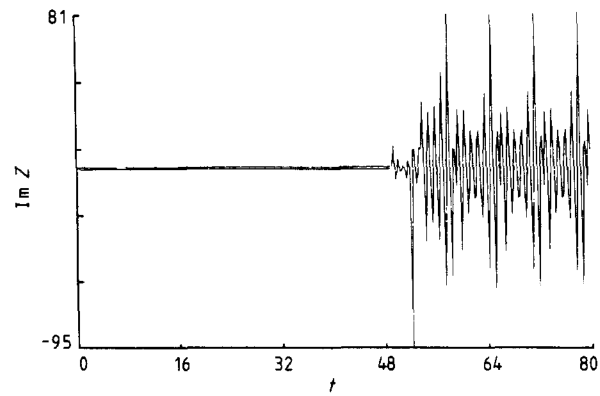

Figure 4. Im Z against time for conjugate initial conditions and $\varepsilon = 0$.

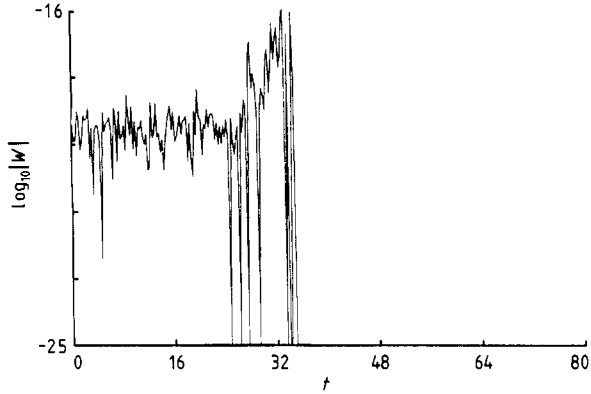

Figure 5. $\mathrm{Log}_{10}\, W$ against time for conjugate initial conditions and $\varepsilon = 0$.

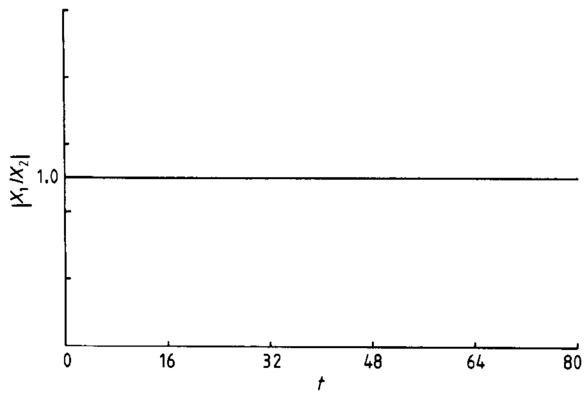

Figure 6. $|X_1/X_2|$ against time for conjugate initial conditions and $\varepsilon = 0$.

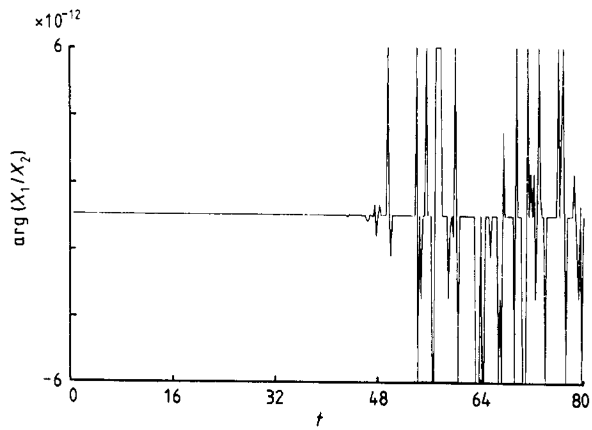

Figure 7. $\mathrm{Arg}(X_1/X_2)$ against time for conjugate initial conditions and $\varepsilon = 0$.

In fact the later time behaviour in figures 1-4 is periodic. This can be seen from figures 8 and 9. We are on a new attractor!

(b) A deterministic case with slightly non-conjugate initial conditions

$$X_1 = X_2 = 0.63$$
$$Y_1 = 0.37 + 10^{-6}\mathrm{i} \qquad (56)$$
$$Y_2 = 0.37 \qquad Z = 9.5.$$

Figure 10 shows a more rapid onset of significant departure from conjugacy.

Again, $\log|W|$, $|X_1|/|X_2|$ and $\arg X_1/X_2$ show the expected behaviour. Moreover, the new attractor is again found to be the limit cycle of (a).

(c) This case involves the noise strength $\varepsilon = 10^{-8}$ but conjugate initial conditions as in (a).

Figure 11 shows the departure from conjugacy which is more severe.

$\mathrm{Log}|W|$ takes values near the noise floor while $|X_1|/|X_2|$ and $\arg X_1/X_2$ show the expected behaviour and remain near 1 and 0 respectively. Noise causes large excursions from the limit cycle attractor of (a).

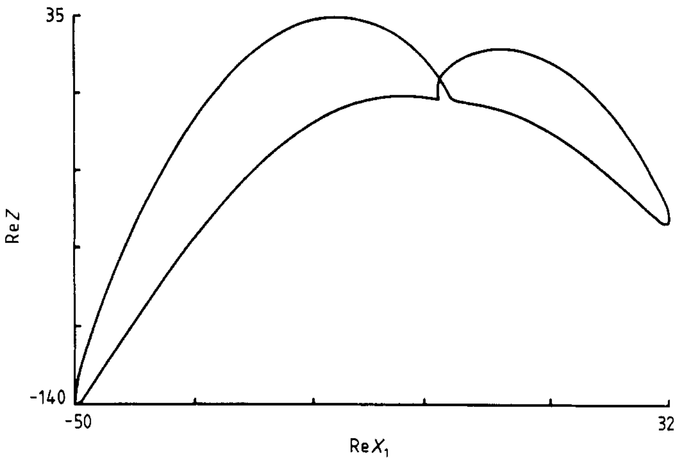

Figure 8. Re Z against Re X_1 for conjugate initial conditions and $\varepsilon = 0$.

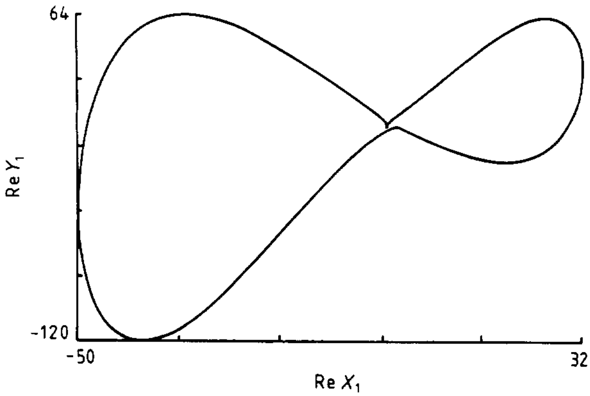

Figure 9. Re Y_1 against Re X_1 for conjugate initial conditions and $\varepsilon = 0$.

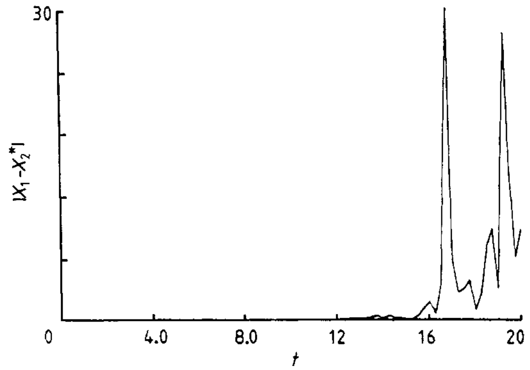

Figure 10. $|X_1 - X_2^*|$ against time for non-conjugate initial conditions and $\varepsilon = 0$.

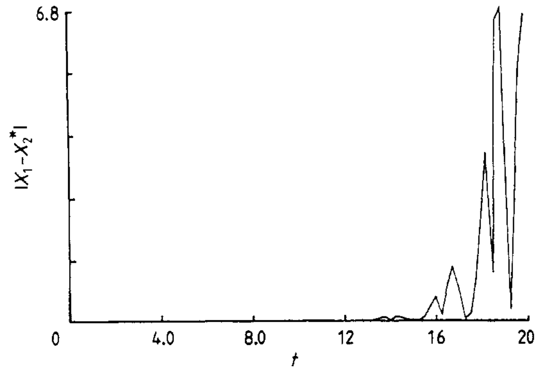

Figure 11. $|X_1 - X_2^*|$ against time for conjugate initial conditions and $\varepsilon = 10^{-8}$.

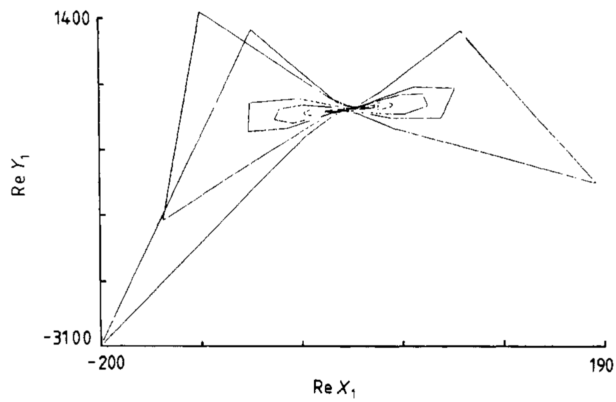

Figure 12. Re Y_1 against Re X_1 for conjugate initial conditions and $\varepsilon = 10^{-3}$.

(d) This case involves a noise strength of $\varepsilon = 10^{-3}$ but conjugate initial conditions. We have tried different initial conditions but there is a 'blow up' of the trajectories. For such noise strengths the calculation involving the Wigner representation [13] was satisfactory. Figure 12 shows typical behaviour and represents 15 time units of evolution.

The way that conjugacy is being broken (e.g. by going on to a new attractor or divergence of trajectories) shows that averaging will not recover behaviour for expectation values in the chaotic region which is compatible with that found using the Wigner representation.

(e) The steady states of (24) and (25) are stable to noise (provided it is not too large, namely $\varepsilon < 10^{-2}$).

We conclude that although the positive P representation is satisfactory in the lasing and sublasing regions, above the chaos threshold the additional dimensions allow extra instabilities. These instabilities are sufficiently severe to drastically alter the nature of the solution from that which is expected.

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Appendix

We shall give a linearised stability analysis for the deterministic positive P Lorenz equations

$$\begin{aligned} &\dot{x}_1 = -\sigma(x_1 - y_1) \qquad &&\dot{x}_2 = -\sigma(x_2 - y_2) \\ &\dot{y}_1 = -y_1 + x_1 z &&\dot{y}_2 = -y_2 + x_2 z \\ &\dot{z} = -b(z-r) - x_1 y_2 - x_2 y_1 . \end{aligned} \tag{A1}$$

We write

$$\begin{aligned} x_1 &= X_1^{(0)} + \mathrm{i}\, Y_1^{(0)} + f_1^x + \mathrm{i} f_1^y \\ x_2 &= X_2^{(0)} + \mathrm{i}\, Y_2^{(0)} + f_2^x + \mathrm{i} f_2^y \\ y_1 &= X_1^{(0)} + \mathrm{i}\, Y_1^{(0)} + p_1^x + \mathrm{i} p_1^y \\ y_2 &= X_2^{(0)} + \mathrm{i}\, Y_2^{(0)} + p_2^x + \mathrm{i} p_2^y \\ z &= Z^{(0)} + z_x + \mathrm{i} z_y \end{aligned} \tag{A2}$$

where $Z^{(0)}$, $X_1^{(0)}$, $X_2^{(0)}$, $Y_1^{(0)}$ and $Y_2^{(0)}$ are real and define the steady state. We define

$$\begin{aligned} &f_\pm^x = \tfrac{1}{2}(f_1^x \pm f_2^x) \qquad &&f_\pm^y = \tfrac{1}{2}(f_1^y \pm f_2^y) \\ &p_\pm^x = \tfrac{1}{2}(p_1^x \pm p_2^x) &&p_\pm^y = \tfrac{1}{2}(p_1^y \pm p_2^y) \\ &X_\pm^{(0)} = \tfrac{1}{2}(X_1^{(0)} \pm X_2^{(0)}) &&Y_\pm^{(0)} = \tfrac{1}{2}(Y_1^{(0)} \pm Y_2^{(0)}). \end{aligned} \tag{A3}$$

(Preservation of conjugacy would imply $f_+^y = p_+^y = f_-^x = p_-^x = 0$.)

We then find

$$\dot{f}^x_\pm = -(f^x_\pm - p^x_\pm) \qquad \dot{f}^y_\pm = -(f^y_\pm - p^y_\pm)$$
$$\dot{p}^x_\pm = -p^x_\pm + Z^{(0)} f^x_\pm + X^{(0)}_\pm z_x - Y^{(0)}_\pm z_y$$
$$\dot{p}^y_\pm = -p^y_\pm + Z^{(0)} f^y_\pm + Y^{(0)}_\pm z_x + X^{(0)}_\pm z_y \tag{A4}$$
$$\dot{z}_x = -bz_x - 2X^{(0)}_+ f^x_+ - 2X^{(0)}_+ p^x_+ + 2X^{(0)}_- f^x_- + 2X^{(0)}_- p^x_- + 2Y^{(0)}_+ f^y_+ + 2Y^{(0)}_+ p^y_+ - 2Y^{(0)}_- f^y_- - 2Y^{(0)}_- p^y_-$$
$$\dot{z}_y = -bz_y - 2Y^{(0)}_+ f^x_+ - 2X^{(0)}_+ p^y_+ + 2Y^{(0)}_- f^x_- + 2X^{(0)}_- p^y_- - 2X^{(0)}_+ f^y_+ - 2Y^{(0)}_+ p^x_+ + 2X^{(0)}_- f^y_- + 2Y^{(0)}_- p^x_- .$$

For the steady state

$$X^{(0)}_\pm = Y^{(0)}_\pm = 0 \qquad Z^{(0)} = r$$

(A4) reduces to

$$\dot{V}_i = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} V_i \qquad i = 1, \ldots, 4 \tag{A5}$$
$$\dot{V}_5 = -bV_5$$

with

$$V_1 = (f^x_+, p^x_+)^T \qquad V_2 = (f^x_-, p^x_-)^T$$
$$V_3 = (f^y_+, p^y_+)^T \qquad V_4 = (f^y_-, p^y_-)^T$$
$$V_5 = (z_x z_y)^T.$$

The eigenvalues corresponding to the system of equations in (A5) are

$$\lambda_\pm = \tfrac{1}{2}(1+\sigma) \pm [\tfrac{1}{4}(1+\sigma)^2 - \sigma(1-r)]^{1/2}$$
$$\lambda_5 = b. \tag{A6}$$

λ_+ becomes positive for $r > 1$ (which is the usual laser threshold). In fact from (A5) we see that there are four different eigenvectors corresponding to λ_+.

Next we consider the stability analysis for the states given in (18)-(20) in § 3. It is useful to make a change of phase so that the steady states for the field and polarisation are real, i.e.

$$Y^{(0)}_\pm = 0. \tag{A7}$$

The equation (A4) decouples into two sets of five equations, namely

$$\dot{V}_{x,y} = MV_{x,y} \tag{A8}$$

where

$$M = \begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0 \\ 0 & -\sigma & 0 & \sigma & 0 \\ 1 & 0 & -1 & 0 & X^{(0)}_+ \\ 0 & 1 & 0 & -1 & X^{(0)}_- \\ -2X^{(0)}_+ & 2X^{(0)}_- & -2X^{(0)}_+ & 2X^{(0)}_- & -b \end{pmatrix} \tag{A9}$$

and

$$V_x = (f_+^x, f_-^x, p_+^x, p_-^x, z_x)^T \tag{A10}$$

$$V_y = (f_+^y, f_-^y, p_+^y, p_-^y, z_y)^T. \tag{A11}$$

The eigenvalues λ of M satisfy

$$\lambda(\lambda+\sigma+1)[\lambda(\lambda+\sigma+1)(\lambda+b)+b(r-1)(\lambda+2\sigma)]. \tag{A12}$$

(A11) is the basis for the eigenvectors referred to in § 3. Apart from the eigenvalues

$$\lambda_1 = 0 \qquad \lambda_2 = -(\sigma+1) \tag{A13}$$

the remaining solutions of (A12) are precisely those that occur in the conventional Lorenz equations.

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